An Embedding-Space Tool for Understanding the Geometry of Gauge Theories

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Teaching Geometry of . . .
Preview
Embedding & Curvature
Embedding Gauge Theory
Magnetic Fields
Electric Fields
Geometry examples
Charge
Acceleration
Conclusion

1. Teaching Geometry of Gauge Theory

- Defined by symmetry group
- Similarities to geometry

Object	General Relativity	Gauge Theory
Covariant Derivative	$\nabla_\sigma V^\alpha = (\delta^\alpha_\mu \partial_\sigma + \Gamma^\alpha_{\sigma\mu}) V^\mu$	$D_\mu \phi^a = (\delta^a_b \partial_\mu - iq A_\mu{}^a{}_b) \phi^b$
Connection Coefficient	$\Gamma^\alpha_{\sigma\mu}$	$-iq A_\mu{}^a{}_b$
Curvature Tensor	$R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma = ([\nabla_\mu, \nabla_\nu] V)^\lambda$	$-i\, F_{\mu\nu}{}^a{}_b \phi^b = ([D_\mu, D_\nu]\phi)^a$

- Visualize the geometry?

2. Preview

- **Visual use of an old tool.**
  - ◇ Narasimhan and Ramanan [1, 2], Atiyah [3], Dubois-Violette and Georgelin [4], Cahill and Raghavan [5]
- **Today's Focus:**
  - **Quantum-mechanics with Electromagnetism**
- **Introduce embedding and curvature**
- **Embed a Gauge Theory**
- **See why:**
  - $\frac{1}{2}\vec{E}^2$ **is "kinetic" energy**
  - $\frac{1}{2}\vec{B}^2$ **is "potential" energy**
- **Geometry examples: solenoid, plane wave**
- **Charge**
- **Acceleration**

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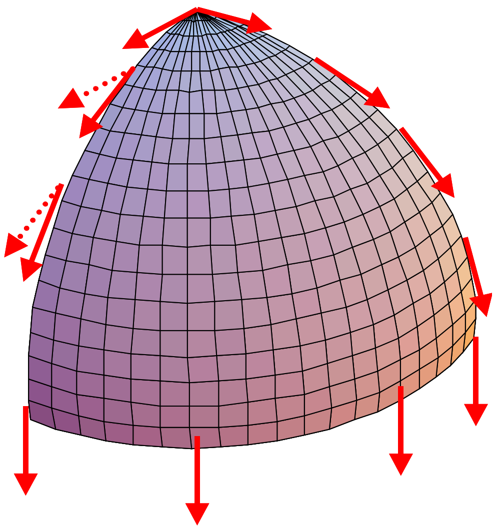
3. Embedding & Curvature

- Embedding makes geometry simple.



- Curvature is $R = \frac{\theta}{\text{Area}} = \frac{\pi/2}{4\pi r^2/8} = \frac{1}{r^2}$

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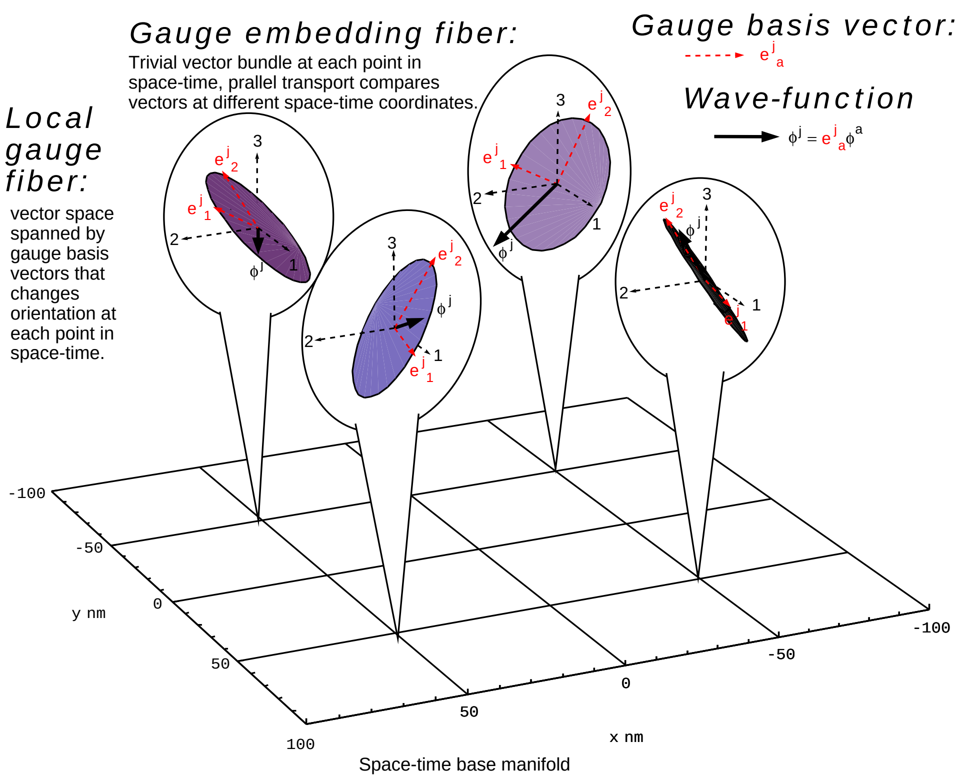
4. Embedding Gauge Theory

- **Embed gauge theory also makes understanding simple.**



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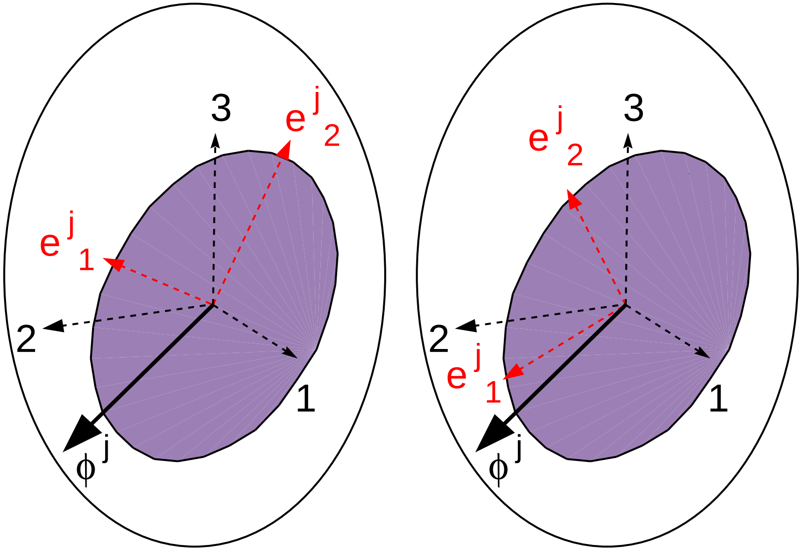
- A close up of the geometrical objects at each space-time point.



Wave-function

→ $\phi^j = e^j{}_a \phi^a$

Gauge basis vector:

- - - → $e^j{}_a$

Local gauge fiber:

Vector space spanned by gauge basis vectors that changes orientation at each point in space-time.

Gauge embedding fiber:

Trivial vector bundle at each point in space-time, prallel transport compares vectors at different space-time coordinates.

- Matter vector field is a wave-function
- Gauge choice is choice of basis vectors

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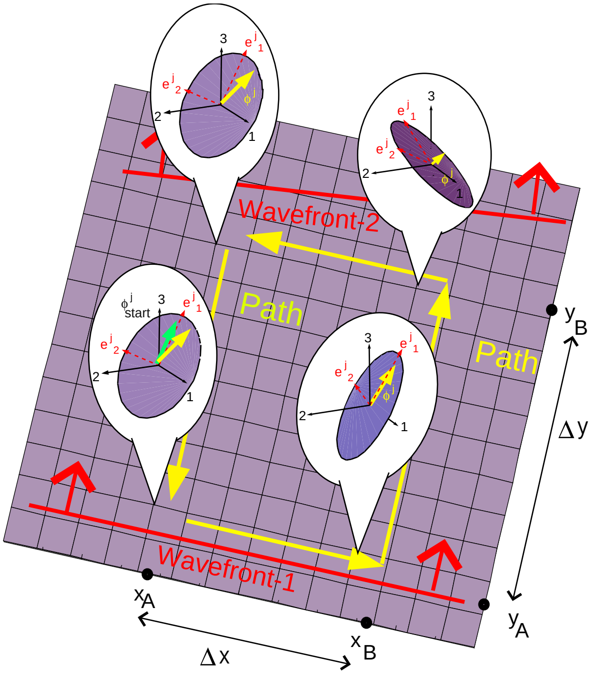
5. Magnetic Fields

- $B_z = \dfrac{\theta}{\Delta x\, \Delta y}$

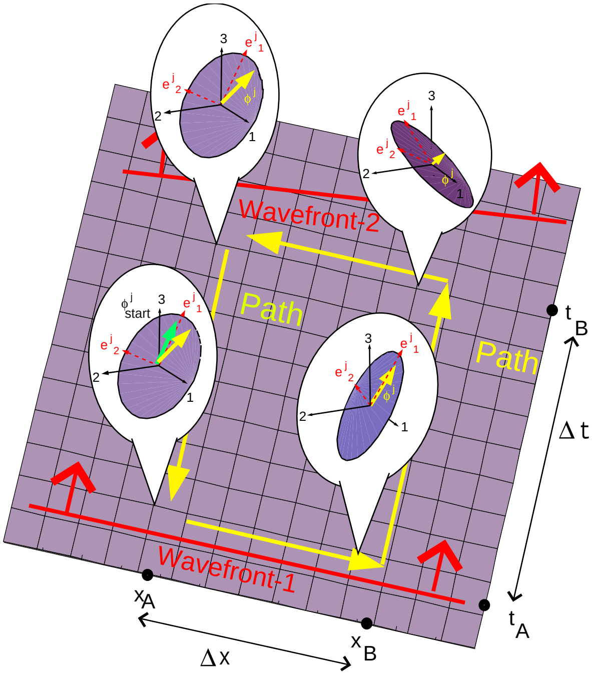


Space-time base manifold

- $\vec{B}$ is a measure of curvature at a fixed time

6. Electric Fields

- $E_x = \dfrac{\theta}{\Delta x\, \Delta t}$



Space-time base manifold

- $\vec{E}$ is a measure of change of the curvature as time goes by.

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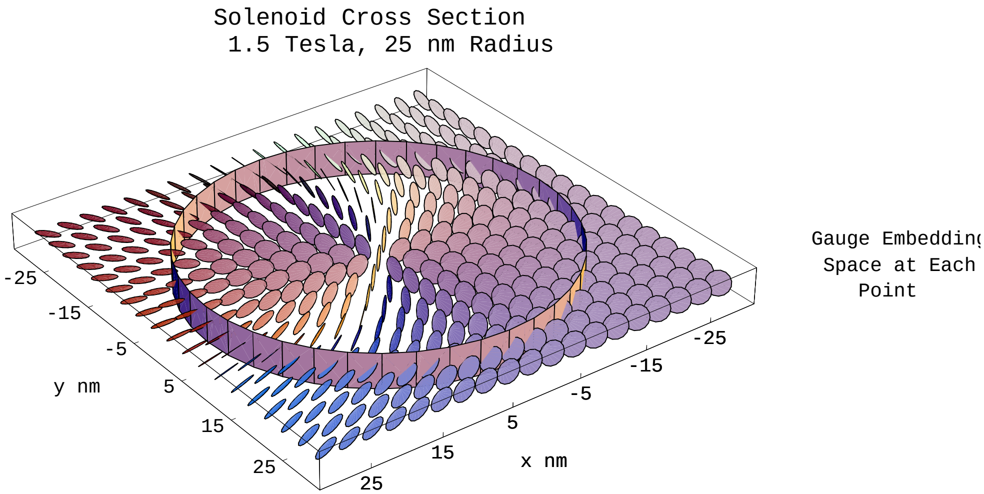
7. Geometry examples

- **Geometry of an infinite solenoid**



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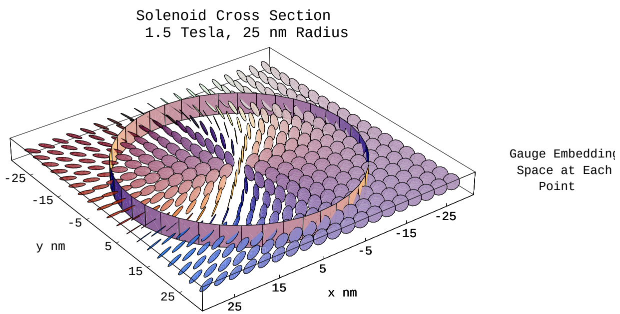
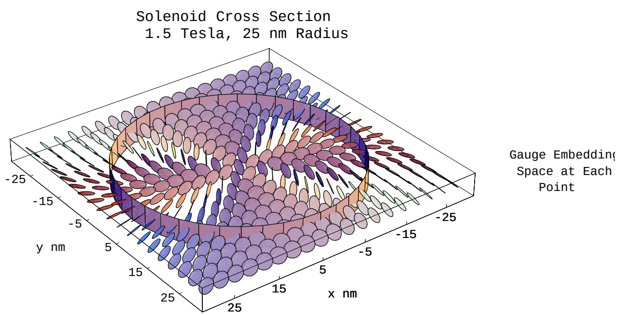
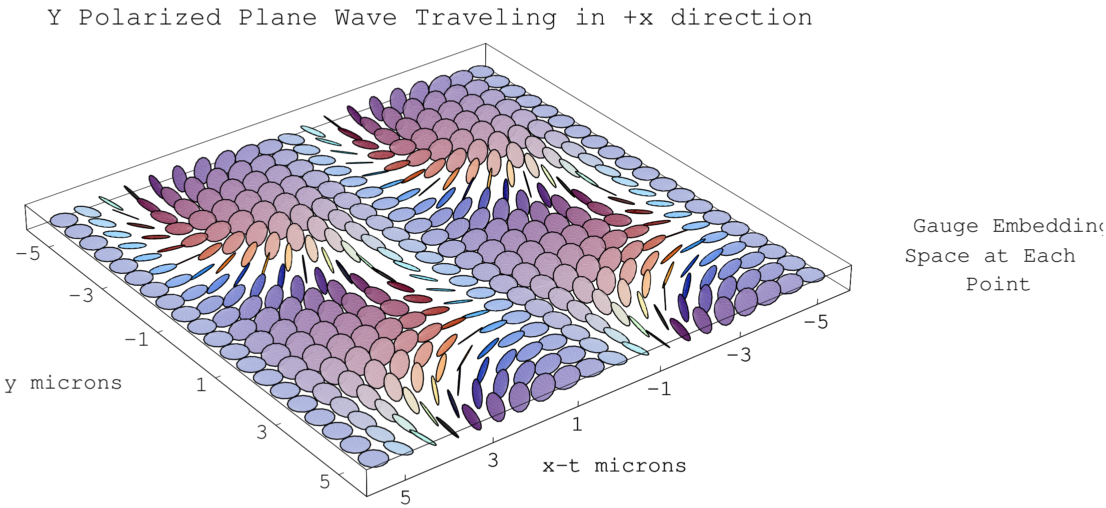
- Disadvantage of embedding
    - Map $\vec{E}$ and $\vec{B}$ to an embedding is not unique.
    - Both diagrams depict the same magnetic field strength





• Geometry of an infinite plane wave



• Show animations

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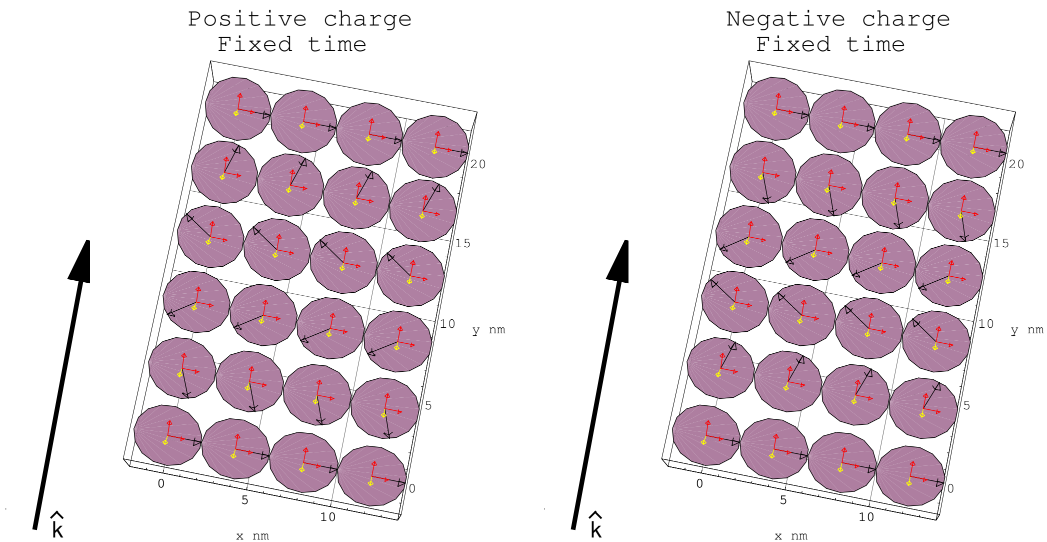
8. Charge

- **Wave-function rotation determines charge**



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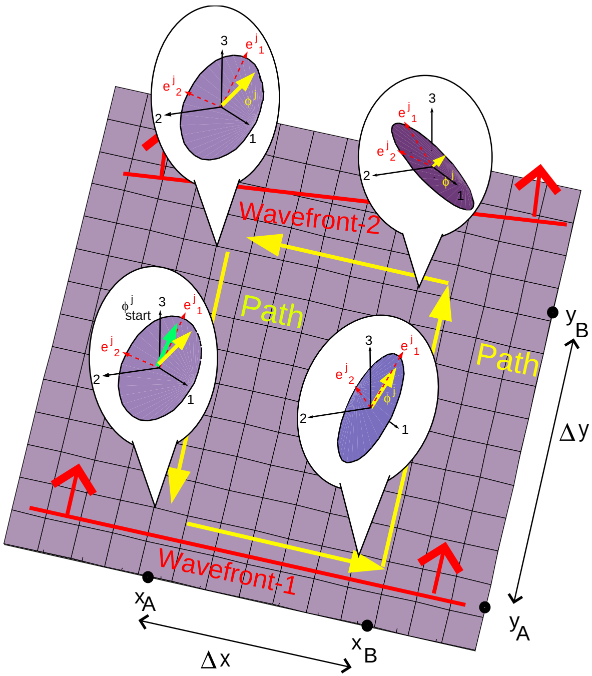
9. Acceleration



Space-time base manifold

- **Imagine negative (CCW) rotating wave-function moving $+\hat{y}$**
- **Curvature aids the rotation on left side of wave-function**
- **Wave-front tilted left**

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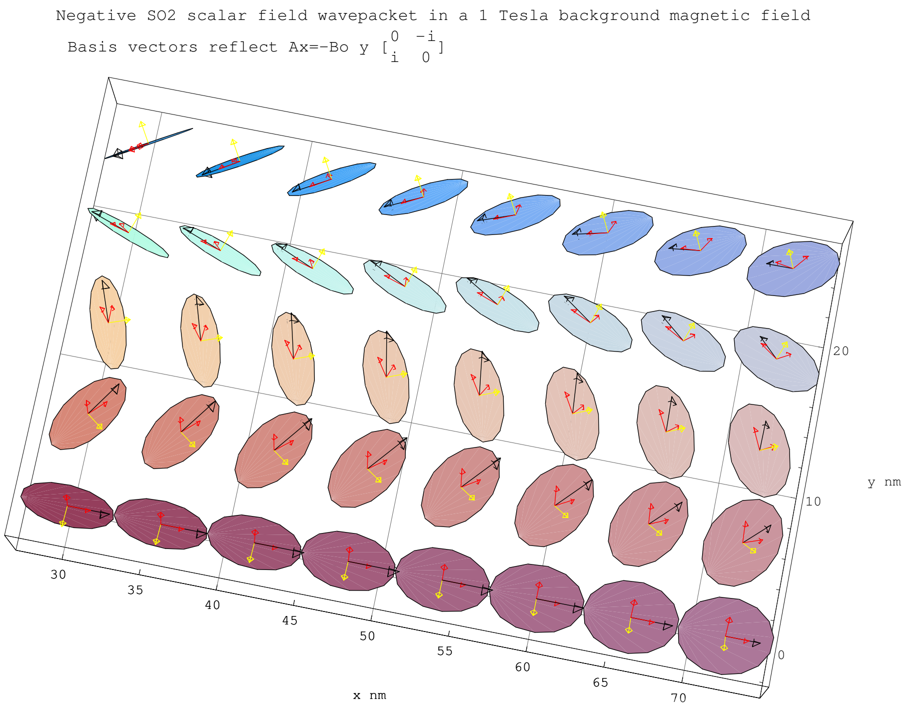
- **Example of negatively charged wave-function in constant $\vec{B}_z$**



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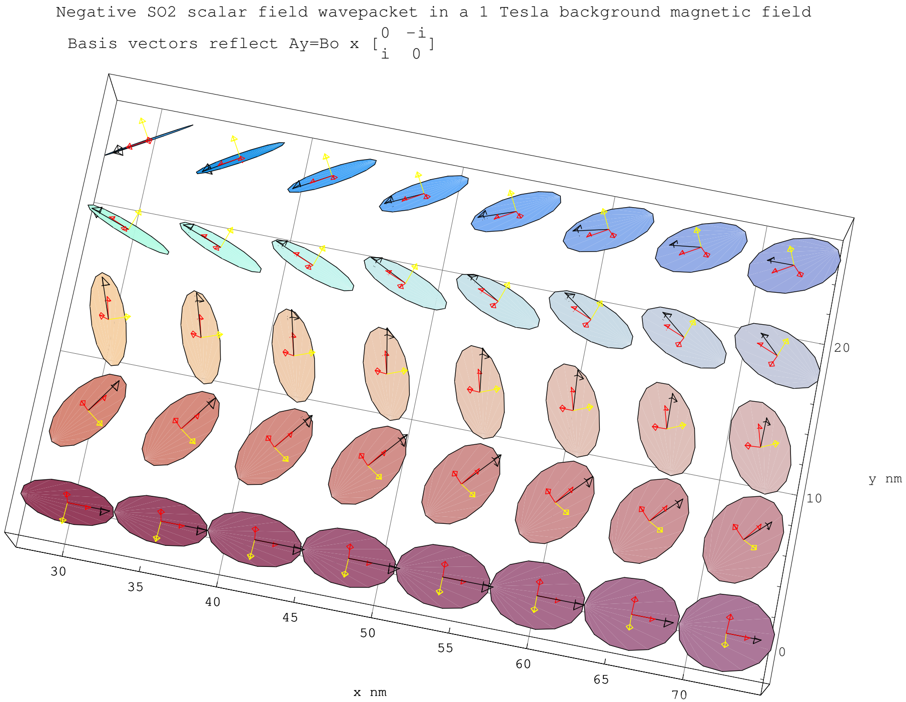
- **Example of negatively charged wave-function in constant $\vec{B}_z$**



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10. Conclusion

- Embedding gauge theory makes geometry simple
- Visualize fundamental components
  - Gauge invariant objects
  - Lorentz invariant objects
  - Symmetry among forces
- Represent vs Define
  - "Represented" gauge theory with geometry
  - "Define" gauge theory geometrically has other interesting consequences.
- For more information: /hep-th/0205250

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References

[1] **M. S. Narasimhan and S. Ramanan,** *Existence of universal connections*, ***Am. J. Math.*** **83 (1961) 563.**

[2] **M. S. Narasiman and S. Ramanan,** *Existence of universal connections ii*, ***Am. J. Math.*** **85 (1963) 223.**

[3] **M. F. Atiyah,** ***Geometry of Yang-Mills Fields (Lezioni Fermiane).*** **Sc. Norm. Sup., Pisa, Italy, 1979.**

[4] **M. Dubois-Violette and Y. Georgelin,** *Gauge theory in terms of projector valued fields*, ***Phys. Lett.*** **B82 (1979) 251.**

[5] **K. Cahill and S. Raghavan,** *Geometrical representations of gauge fields*, ***J. Phys.*** **A26 (1993) 7213–7217.**